

T1: Permanent magnets**Part A: Interaction of two magnets.**

(a) Since the distance between the magnets is big as compared to their size, we can approximate each of them as a dipole of magnitude

$$m = \frac{\pi}{4} d^2 h J = 0.75 \text{ A m}^2.$$

Since the two dipoles are parallel to each other and to the line connecting them, the interaction energy of one of the magnets with the field of the other magnet is

$$W = -\vec{B} \cdot \vec{m} = -\frac{\mu_0 m^2}{2\pi L^3},$$

and by taking a derivative we obtain

$$F_1 = \frac{dW}{dL} = \frac{3\mu_0 m^2}{2\pi L^4} \approx 0.21 \text{ mN}.$$

a) $m = \frac{\pi}{4} d^2 h J$	0.2 pts
b) $W = -\vec{B} \cdot \vec{m}$	0.1 pts
c) $F_1 = \frac{dW}{dL}$	0.1 pts
d) $F_1 = \frac{3\mu_0 m^2}{2\pi L^4}$	0.1 pts
Answer: 0.21 mN	0.1 pts

(b) In the case of a homogeneous magnetization, the molecular currents in the bulk of the material cancel out, leaving only a surface current at the surfaces which are not perpendicular to the magnetization vector. Hence, there is a surface current on the side surfaces of the cylinder. As the height of the surface is much smaller than the radius, these currents can be approximated as a ring current I ; the dipole moment $\pi d^2 I/4$ of the ring current must be equal to the total dipole moment $\pi d^2 h J/4$ of the magnet, hence $I = Jh \approx 2.4 \text{ kA}$.

a) currents inside cancel out, surface currents remain	0.1 pts
c) $m = I A$	0.1 pts
d) $I = Jh$	0.1 pts
Answer: 2.4 kA	0.1 pts

(c) Since the distance between the magnets is now significantly smaller than their diameter, the force can be approximately found as the force between two straight currents I of length πd at distance L :

$$F_2 = \frac{\mu_0 I^2}{2\pi L} \pi d = \frac{\mu_0 I^2 d}{2L} \approx 14 \text{ N}.$$

a) Consider as straight currents	0.3 pts
b) $B = \frac{\mu_0 I}{2\pi L}$	0.3 pts
c) $F_2 = \pi d I B$	0.2 pts
d) $F_2 = \frac{\mu_0 I^2 d}{2L}$	0.1 pts
Answer: 14 N	0.1 pts

(d) The chain will most likely break below the top-most magnet because then the magnetic pull between the magnets needs to compensate the largest possible weight. Let the number of magnets be $N + 1$, and the mass of a single magnet $M = \frac{\pi}{6} \rho \delta^3 \approx 0.5 \text{ g}$; then the

weight of the magnets $F = MNg$ is balanced by the magnetic force

$$F = \frac{3\mu_0 m^2}{2\pi \delta^4} \sum_{n=1}^N \frac{1}{n^4} = \mu_0 m^2 \pi^3 60 \delta^4,$$

where $m = \frac{\pi}{6} J \delta^3 \approx 78 \text{ mA m}^2$ and we have assumed that $N \gg 1$ so that we can assume in the sum $N = \infty$. From the force balance we obtain

$$N = \frac{\mu_0 m^2 \pi^3}{60 M g \delta^4} \approx 1320;$$

hence, the total length of the chain is $N\delta = 6.6 \text{ m}$. Note that $N = 1320$ is indeed much bigger than 1.

a) It will break at the top	0.1 pts
b) $M = \frac{\pi}{6} \rho \delta^3$	0.1 pts
c) $F = MNg$	0.2 pts
d) $F = \frac{3\mu_0 m^2}{2\pi \delta^4} \sum_{n=1}^N \frac{1}{n^4}$	0.2 pts
e) $m = \frac{\pi}{6} J \delta^3$	0.2 pts
f) $N = \frac{\mu_0 m^2 \pi^3}{60 M g \delta^4}$	0.1 pts
Answer: 1260	0.1 pts

Remark: if the sum is substituted with a finite sum as an approximation, with two or three terms in it, full marks are given. If only one term is kept, subtract 0.1 from d) or f).

(e) Each of the balls creates magnetic field of a dipole m ; the magnetic dipole creates the same field which would be created by two magnetic charges, equal by modulus to q and of opposite sign, at a distance $s = m/q$, assuming that this distance s is much smaller than the distance from the dipole to the observation point. Here it is convenient to select $s = \delta$ (hence $q = m/\delta$) because in that case almost all the positive and negative magnetic charges overlap and cancel out each other. The only ones which will not cancel out are the magnetic charges at the chain's endpoints. One of these charges is very far so that the field at P is the field of a magnetic charge at O :

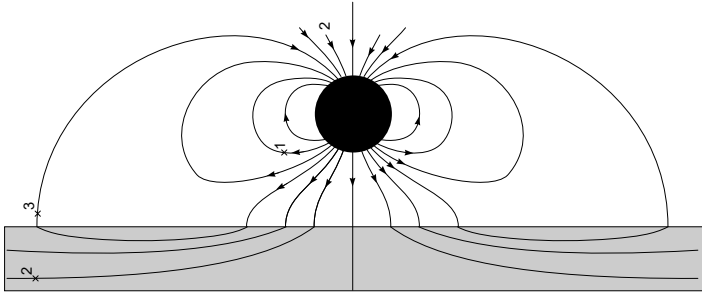
$$B = \frac{\mu_0 q}{4\pi r^2} = \frac{\mu_0 m}{4\pi \delta r^2} = \frac{J \mu_0 \delta^2}{24 r^2}.$$

a) Idea of magnetic charges	0.4 pts
b) $q = m/\delta$	0.4 pts
c) $B = \frac{\mu_0 q}{4\pi r^2}$	0.4 pts
d) $B = \frac{J \mu_0 \delta^2}{24 r^2}$	0.3 pts

Part B: Interaction of magnets with ferromagnetic materials.

(a) Due to the boundary condition at the surface of the ferromagnet, the field lines must enter the plates almost perpendicularly. Indeed, as it follows from the Ampère's circuital law, the tangential component of \vec{B}/μ is continuous at the surface of a ferromagnet; similarly, the Gauss law for the magnetic field implies that the normal component of the B -field is continuous. From these two facts, one can derive the "refraction law" for the field lines, $\tan \alpha = \mu \tan \beta$, where α and β are the angles between the tangents of a field line and the surface normal, inside

and outside of the ferromagnetic, respectively. From the fact that $\mu \gg 1$ we can deduce that as long as α is not small, $\beta \approx \pi/2$. Those field lines which enter the plate must exit it somewhere, this happens somewhere farther away from the magnet, see the sketch below.



- | | |
|-------------------------|---------|
| a) Field line 1 correct | 0.2 pts |
| b) Field line 2 correct | 0.4 pts |
| c) Field line 3 correct | 0.4 pts |

Remarks:

- i) Subtract 0.1 both from b) and c) if the field line does not enter the plate perpendicularly;
- ii) Subtract 0.1 both from b) and c) if the field line does not refract;
- iii) Subtract 0.1 both from b) if its segment rightwards of the magnet is not shown;
- iv) Subtract 0.1 both from a) and c) if the field line does not form a closed loop.

(b) The problem can be solved by introducing an image magnet — a mirror reflection of the real magnet with respect to the surface of the plate, with the dipole moment being both reflected and flipped. With this image magnet, the boundary condition above the plate is satisfied: the field lines enter the plate perpendicularly. Hence, the force and torque exerted to the real magnet are equal to the force and torque exerted by the image magnet. The equilibrium is achieved when the dipole is parallel to the field created by the image magnet which is the case when the dipole moment is perpendicular to the plate. The force is almost the same as what was already found in part A(d), with the only difference that now there is only the first term in the sum:

$$F = \frac{3\mu_0 m^2}{2\pi\delta^4} = 5.9 \text{ N.}$$

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|---|----------|
| a) Idea of magnetic image | 0.3 pts |
| b) Correct direction of the image \vec{J} | 0.2 pts |
| c) $F = \frac{3\mu_0 m^2}{2\pi\delta^4}$ | 0.2 pts |
| d) $F = 5.9 \text{ N}$ | 0.1 pts |
| e) each correct tick | 0.1 pts |
| f) each incorrect tick | -0.1 pts |

Remark: if the sum of d) and e) is negative, e) is taken and minus d) yielding 0 in total.

(c) As explained above, the magnetic field lines are perpendicular to the surface of the ferromagnetic plate. Since the gap is narrow as compared to its width, the field lines are inside the gap almost straight. Due to the Ampère's circulation theorem it also means that the field in the gap is homogeneous. Due to the Ampère's circulation theorem, field outside the gap vanishes as the gap's

width tends to 0, so in the limit all flux through the permanent magnet wraps around through the gap; see the sketch of magnetic field lines. Now, let us recall that the disc magnet is equivalent to a surface current of density J along the curved surface of the disc. Hence we can write the circulation theorem along the loop defined by one of the field lines shown in the figure:

$$I = \oint \vec{H} \cdot d\vec{r} \approx (B_1 + B_2)h/\mu_0,$$

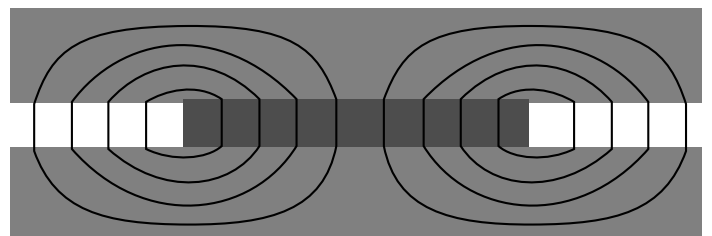
where B_1 and B_2 denote the flux density inside the permanent magnet and outside the magnet (but still inside the slit), respectively. Here we have neglected the contribution of the magnetic field inside the ferromagnetic plate to the integral because μ is very big. Due to the Gauss law, $\frac{\pi}{4}d^2 B_1 = \frac{\pi}{4}(D^2 - d^2)B_2$; with $D = 2d$ this yields $B_1 = 3B_2$. Thus, $B_2 = I\mu_0/4h = J\mu_0/4 = 0.375 \text{ T}$ and $B_1 = 1.125 \text{ T}$. In order to find the force exerted to one of the ferromagnetic plates, we can notice that the force does not depend on what is creating the magnetic field and, hence, we can substitute the disc magnet with the current I in a superconducting ring. Next we apply the virtual displacement method and increase the distance between the plates by dx . In the case of a superconducting ring, the magnetic flux through the ring is conserved, and therefore, the magnetic field strength inside the gap will remain unchanged during the virtual displacement. With all this information we are ready to calculate the change of the magnetic field energy. The magnetic field energy inside the ferromagnet can be neglected because its density is ca μ times smaller than inside the gap. So, the energy is changed only because the volume of the gap is changed:

$$dW = \frac{\pi}{8\mu_0}[d^2 B_1^2 + (D^2 - d^2)B_2^2]dx = \frac{3\pi}{2\mu_0}B_2^2 d^2 dx$$

which means that the force

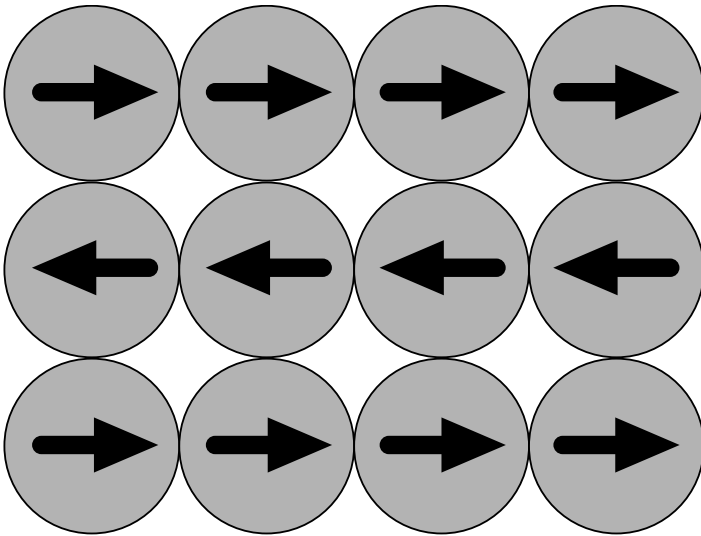
$$F = \frac{dW}{dx} = \frac{3\pi}{2\mu_0}B_2^2 d^2 = \frac{3\pi}{32}J^2\mu_0 d^2 \approx 210 \text{ N.}$$

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|--|---------|
| a) \vec{B} in the slit is homogeneous | 0.2 pts |
| b) \vec{B} in the permanent magnet is homog. | 0.2 pts |
| c) \vec{B} in slit and in perm. magn. is normal | 0.1 pts |
| e) $I = (B_1 + B_2)h/\mu_0$ | 0.2 pts |
| f) $\frac{\pi}{4}d^2 B_1 = \frac{\pi}{4}(D^2 - d^2)B_2$ | 0.2 pts |
| g) $B_2 = I\mu_0/4h$ | 0.1 pts |
| h) $B_1 = 3I\mu_0/4h$ | 0.1 pts |
| i) $dW = \frac{\pi}{8\mu_0}[d^2 B_1^2 + (D^2 - d^2)B_2^2]dx$ | 0.1 pts |
| j) $F = \frac{dW}{dx}$ | 0.1 pts |
| k) $\frac{3\pi}{32}J^2\mu_0 d^2$ | 0.1 pts |
| l) $F \approx 210 \text{ N.}$ | 0.1 pts |



Part C: Model of ferromagnetic and anti-ferromagnetic materials.

(a) Since the task is about finding only one configuration of dipoles, we can just try looking for configurations satisfying the requirements. The simplest approach is to start construction with the chain of magnets described in part A(d): if all the dipoles are directed parallel to each other and parallel to the chain, the system is obviously in equilibrium. Now, two such chains can be parallel to each other, and they can be also antiparallel. In both cases, each of the balls is in a stable equilibrium in terms of rotations. Indeed, each of the balls from the left and from the right contribute the field $\vec{B}_1 = \hat{x} \frac{\mu_0 m}{2\pi\delta^3}$, while each of the balls from above and below contribute $\vec{B}_2 = \pm \frac{1}{2} \vec{B}_1$, where \hat{x} denotes a horizontal unit vector; '+' corresponds to antiparallel rows, and '-' — to parallel rows. Since $B_2 < B_1$, the sum of the four contributions is always pointing in the direction of \hat{x} which ensures the rotational stability of the magnet. Attraction force between two neighbouring rows is contributed only by the vertical nearest-neighbour pairs of balls, so we can just calculate only the interaction force between two such magnets. If two such balls were to be at distance y , the interaction energy would be $W = \pm \frac{\mu_0 m^2}{4\pi y^3}$ so that the y -directional force $F_y = \frac{dW}{dy} = \mp \frac{3\mu_0 m^2}{4\pi y^4}$. This means that the two balls attract if they are antiparallel and repel otherwise. This brings us to the conclusion that the order must be antiferromagnetic, shown below in the sketch.

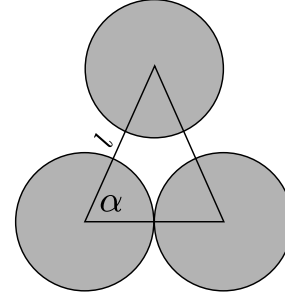


The work needed to pull out one of the magnets is easily found as its interaction energy with the four nearest neighbours, with minus sign, i.e. $W = \vec{B} \cdot \vec{m}$, where $\vec{B} = 2\vec{B}_1 + 2\vec{B}_2 = \frac{3\mu_0 m}{2\pi\delta^3} \hat{x}$ so that $W = \frac{3\mu_0 m^2}{2\pi\delta^3} = 29 \text{ mJ}$.

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|--|---------|
| a) Fig: left and right parallel magnets attract | 0.1 pts |
| b) Fig: top and bottom antipar. magn. attract | 0.1 pts |
| c) \vec{B} from the 4 neighbours $\parallel \vec{m} \Rightarrow$ no torque | 0.1 pts |
| d) correctly marked 12 arrows | 0.1 pts |
| e) antiferromagnetic | 0.1 pts |
| f) $W = \vec{B} \cdot \vec{m}$ | 0.1 pts |
| g) $W = \frac{3\mu_0 m^2}{2\pi\delta^3}$ | 0.1 pts |
| h) $W = 29 \text{ mJ}$ | 0.1 pts |

Remark: no marks for d) if any of the magnets has a wrong direction or has no arrow.

(b) Now we need to repeat the steps done for the previous question, with the only difference in the mutual placement of the magnets. Also, each of the magnets of the top row interacts now with two magnets of the bottom row with the three magnets forming an equilateral triangle. Since we'll be going to use virtual displacement method, we consider the interaction of three magnets forming an isosceles triangle as shown in the figure; while the base of the triangle remains fixed during virtual displacements, the length of the sides l will change.



First we need an expression of the magnetic field caused by the two bottom magnets at the centre of the topmost magnet. Due to symmetry, this field must be horizontal; we can use the formula provided in the problem text for finding it. The dipole moment of the left-bottom magnet needs to be divided into components parallel and perpendicular to the radius vector drawn from its centre to the centre of the topmost magnet, $m_{\parallel} = m \cos \alpha$ and $m_{\perp} = m \sin \alpha$. Hence, we can express the resultant x -component of the magnetic field as

$$\vec{B}_{3x} = \frac{\mu_0}{4\pi l^3} (2\vec{m}_{\parallel} \cos \alpha - \vec{m}_{\perp} \sin \alpha) = \frac{\mu_0 \vec{m}}{4\pi l^3} (3 \cos^2 \alpha - 1).$$

The magnetic field due to both magnets is therefore $2B_{3x}\hat{x}$.

As the first thing, we can now analyse the stability of a magnet with respect to rotations. As before, we assume that the magnets in one single row are parallel to each other, and the magnets at the two neighbouring rows are either parallel or antiparallel to each other. In either case, the rows at the top and at the bottom from a given magnet are parallel to each other; let them be oriented along \hat{x} . Then, each row contributes $2B_{3x}\hat{x}$ to the total field at the position of our magnet. The total field has also contributions $\vec{B}_{4x} = \pm \frac{\mu_0 m}{2\pi\delta^3}$ from the left and right magnets; here '+' corresponds to the ferromagnetic order, and '-' — to the antiferromagnetic order. Keeping in mind that $l = \delta$ and $\cos \alpha = \frac{1}{2}$ the total field is

$$\vec{B}_5 = 4\vec{B}_{3x} + 2\vec{B}_{4x} = \frac{\mu_0 m}{2\pi\delta^3} \left(-\frac{1}{2} \pm 2 \right) \hat{x}.$$

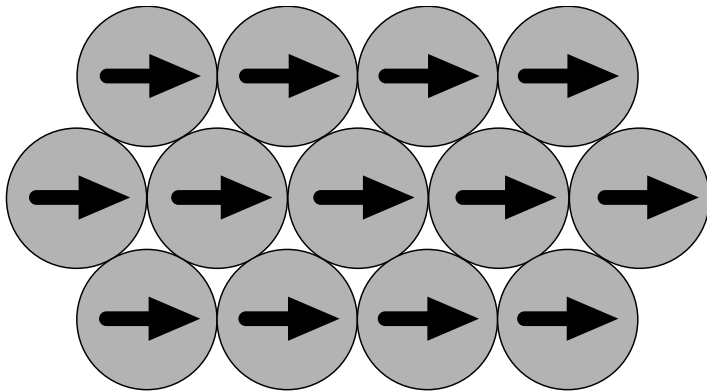
This is parallel to the given magnetic dipole for both '+' and '-', which ensures stability in any case.

With $\vec{m} = \pm \hat{x}m$ and y denoting the height of the isosceles triangle, the vertical component of the interaction force of a magnet with a magnet in the bottom row can be found as

$$F_{5y} = \frac{d}{dy} \vec{B}_3 \cdot \vec{m} = \pm \frac{dl}{dy} \frac{d}{dl} \frac{\mu_0 m^2}{4\pi l^3} \left(\frac{3\delta^2}{4l^2} - 1 \right) = \mp \frac{dl}{dy} \frac{3\mu_0 m^2}{16\pi\delta^3};$$

here we have used $\cos \alpha = \frac{\delta}{2l}$ and upon taking derivative, substituted $l = \delta$. For this force to be attractive, we

need a minus sign which corresponds to the ferromagnetic order (keep in mind that $\frac{dl}{dy} > 0$). Now we are ready to mark the direction of the dipoles on the sketch, see the figure below.



The work needed to pull out a magnet is found similarly to the part (a):

$$W = \vec{B}_5 \cdot m\hat{x} = \frac{3\mu_0 m^2}{4\pi\delta^3} = 15 \text{ mJ}.$$

- | | |
|---|---------|
| a) $\vec{B}_{3x} = \frac{\mu_0 \vec{m}}{4\pi l^3} (3 \cos^2 \alpha - 1)$ | 0.2 pts |
| b) $\vec{B}_{4x} = \pm \frac{\mu_0}{2\pi\delta^3}$ | 0.1 pts |
| c) $\vec{B}_5 = 4\vec{B}_{3x} + 2\vec{B}_{4x}$ | 0.1 pts |
| d) $\vec{B}_5 = \frac{\mu_0 m}{2\pi\delta^3} \left(-\frac{1}{2} \pm 2\right) \hat{x}$ | 0.1 pts |
| e) $F_{5y} = \frac{d}{dy} \vec{B}_3 \cdot \vec{m}$ | 0.1 pts |
| f) $F_{5y} = \mp \frac{dl}{dy} \frac{3\mu_0 m^2}{16\pi\delta^3}$ | 0.1 pts |
| g) F_{5y} attractive | 0.1 pts |
| d) correctly marked 12 arrows | 0.1 pts |
| e) ferromagnetic | 0.1 pts |
| g) $W = \frac{3\mu_0 m^2}{4\pi\delta^3}$ | 0.1 pts |
| h) $W = 15 \text{ mJ}$ | 0.1 pts |

Remark: \pm signes are not required as long as the correct sign corresponding to the ferromagnetic order are used.